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Verification and Validation of the Fractal Attrition Equation

Nigel Perry

Defence Systems Analysis Division
Defence Science and Technology Organisation

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ABSTRACT

Under the auspices of The Technical Cooperation Program (TTCP) Joint Systems & Analysis (JSA) Technical Panel 3 (Joint Concepts and Analysis) research program, researchers in New Zealand and the United Kingdom have been collaborating on the development of a new combat attrition equation. This report examines this new approach, its relationships to existing systems of Lanchester equations and its prospects for further development.

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Executive Summary

Under the auspices of The Technical Cooperation Program (TTCP) Joint Systems & Analysis (JSA) Technical Panel 3 (Joint Concepts and Analysis) research program, researchers in New Zealand and the United Kingdom have been collaborating on the development of a new combat attrition equation. This equation differs from the standard approach developed by Lanchester, as it includes spatial resolution explicitly in its formulation. It was hoped that this would allow for a better representation of the impact of command and control and networked forces in attrition algorithms.

The present work examines this new approach, its relationships to existing systems of Lanchester equations and its prospects for further development.

It concludes that the Fractal Attrition Equation (FAE) was derived using the assumptions and constraints of the MANA simulation and that it does produce results that are consistent with that simulation. In effect, the FAE is verified as a metamodel for MANA. Furthermore, the fractal spatial distribution for each side's forces is shown to result in a fractal temporal distribution for casualties, in keeping with much of the available historical data. However, this novel approach to incorporating the spatial and temporal resolution into a single *force-on-force* equation does not represent a radical departure from traditional practice. Lanchester himself proposed a manual spatial and temporal discretisation, leading to much the same results.

The FAE is shown to be identical to a stochastic Lanchester equation formulation. Lastly, it is shown that the FAE equation of state was derived using mutually exclusive constraints on the relationship between the spatial and temporal resolution applied to the battlefield. A corrected equation of state is derived, using a consistent interpretation of that relationship which is shown to be consistent with those obtained from both deterministic and stochastic Lanchester equation approaches. This was accepted at the TTCP TSA TP3 workshop held at Fishermans Bend in November 2005.

Author

Nigel Perry
Defence Systems Analysis Division

Dr Perry completed his B. Sc. (Hons.) at Melbourne University in 1980 and his Ph. D. at Monash University in 1985. From 1985 to 1989 he was a Research Fellow at Oxford University. From 1989 to 1991 he was a Research Fellow at Monash University. Between 1991 and 1996 he was a lecturer at Victoria University of Technology. He joined DSTO's Maritime Operations Division in 1996 and the Theatre Operations Branch in 1998. He is now a Senior Research Scientist in the Defence Systems Analysis Division

Acronyms, Abbreviations and Symbols

a	Model Outcome
a_i	Independent model variables
A	Battlefield area
A_i	Co-ordinate transformation matrix
$A(x,y)$	Infinitesimal generator matrix between initial state x and final state y
$A(t)$	Strength of arbitrary force at time t
b_i	Dependent model variables
$B(t)$	Strength of Blue force at time t
$B_j(t)$	Strength of Blue force of type j at time t
B_0	Initial strength of Blue force
$C..H$	Arbitrary Exponents
C4ISR	Command, Control, Communications, Computers, Intelligence, Surveillance and Reconnaisance
d	Discrete space interval
dB/dt	Rate of change of Blue force strength with time
dR/dt	Rate of change of Red force strength with time
$E=N^2$	Markov state space of integer pairs
$E[\ ]$	Expectation value
f	Frequency
$f(\ )$	Arbitrary function
FAE	Fractal Attrition equation
$G(t)$	Time dependence of attrition
$I(x,y)$	Unitary matrix
JSA	Joint Systems and Analysis
k_i	Kill rate of force i
k_B	Kill rate of Red force by Blue force
k_R	Kill rate of Blue force by Red force
k_{BR}	Kill rate of Red force by Blue force per Red target
k_{RB}	Kill rate of Blue force by Red force per Blue target
$k_B(i,j)$	Kill rate of Red force of type j by Blue force of type i
$k_R(i,j)$	Kill rate of Blue force of type j by Red force of type i
$\log_d$	Logarithm to the base d
ln	Natural logarithm
MANA	Map Aware Non-uniform Automata
N	Number of boxes
n_d	Average cluster size
ORBSM	Oak Ridge Spreadsheet Model
p_i	Single shot probability of kill of force i
$P_t(x,y)$	Probability that initial state x will transition to final state y at time t
q_B	Reinforcement rate for Blue force
q_R	Reinforcement rate for Red force
QJM	Quantified Judgement Model
r_i	Mean time between detection of force i

$R(t)$	Strength of Red force at time t
$R_j(t)$	Strength of Red force of type *j* at time t
R_0	Initial strength of Red force
TTCP	The Technical Cooperation Program
v	Velocity
$X=(X_t)$	Regular continuous time Markov process
α, β, D, Q	Fractal dimensions
Δt	Discrete time interval
ΔB	Discrete Blue force loss
Σ	Summation over the specified parameters
$\varphi(x,t)$	Distribution of clusters in space and time
$\Phi(\ )$	Arbitrary dimensionless function
Π_j	Similarity Parameters
τ	Arbitrary time parameter
$\Im(\ )$	Fourier Transform
$[x]$	Dimension of variable *x*
$\langle\ \rangle$	Expectation value

Contents

1. Introduction

A theory for success in battle has long been sought, arguably since the time of Sun Tze half a millennium before the Christian era. Since at least the early Twentieth Century, a quantifiable and quantitative expression of such a theory, or model, has been desired. While no longer regarded as providing an accurate prediction on the precise outcome of a battle, such models are used by analysts to provide information to decision makers, who base acquisition decisions involving billions of dollars on them. Thus research into such mathematical models is ongoing.

Researchers in New Zealand and the United Kingdom have been collaborating closely on the development of a new attrition equation for warfare [1]. This equation is different from the standard approach developed by Lanchester because it incorporates spatial patterns into the analysis. It is hoped that this in turn may allow for the better representation of the value of command and control and networked forces in attrition algorithms. They envisaged that this work could lead to the replacement of many ideas in existing aggregated combat models. This work has become part of a research program under The Technical Cooperation Program (TTCP) Joint Systems & Analysis (JSA) Technical Panel 3 (Joint Concepts and Analysis), with an initial report published in July 2005 [1].

The present work reviews this new approach and its relationships to existing systems of Lanchester equations. Consequently, it is recommended that the reader be familiar with the details of this new approach [1] to fully understand the work presented in sections 5 and 6 below. It is considered at three distinct levels; its theoretical derivation and implications, its ability to describe the behaviour of the Map Aware Non-uniform Automata (MANA) cellular automata simulation and its ability to describe the available historical data describing observed battle outcomes.

Prior to describing and addressing those issues, it is considered necessary to briefly review the theory behind Lanchester's equations, extensions to Lanchester's ideas on combat modelling and the theory justifying the application of the metamodelling approach in combat modelling.

2. Lanchester's Equations

During the First World War F. W. Lanchester described one of the simplest, and most enduring, mathematical attrition models of force-on-force combat [2]. He proposed two systems of equations, depending on whether the fighting was "collective" or not. Collective combat between a Red side of strength R and a Blue side of strength B being described by the equations:

$$\frac{dR}{dt} = -k_B B(t), \quad R(0) = R_0$$
$$\frac{dB}{dt} = -k_R R(t), \quad B(0) = B_0 \tag{1}$$

Which result in the equation of state:

$$\frac{\left(R_0^2 - R^2\right)}{\left(B_0^2 - B^2\right)} = \frac{k_B}{k_R} \tag{2}$$

The quadratic form of which results in the system of equations known as the Lanchester Square Law.

Individual combat on the other hand is described by the equations:

$$\frac{dR}{dt} = -k_{BR} B(t) R(t), \quad R(0) = R_0$$
$$\frac{dB}{dt} = -k_{RB} R(t) B(t), \quad B(0) = B_0 \tag{3}$$

With its equation of state:

$$\frac{\left(R_0 - R\right)}{\left(B_0 - B\right)} = \frac{k_{BR}}{k_{RB}} \tag{4}$$

The linear form of which results in the system of equations known as the Lanchester Linear Law.

Over the intervening years there have been many attempts to validate the use of Lanchester's equations to describe combat outcomes through analysis of historical data [3]. These efforts have largely been unsuccessful. Part of the reason for this lack of success is the failure of subsequent users of Lanchester's theory to understand its inherent assumptions, constraints and limitations. Typically, most users assume that the fighting will be entirely collective and that both sides will attrite each other evenly, with strengths asymptotically approaching zero. Yet random chance is a factor in warfare whose effect can be modelled by treating attrition as a stochastic process. This will be covered in greater detail below. Moreover, the Lanchester Square Law equations are unstable and any random deviation from its expectation value will be amplified. The Square Law system of equations for force ***A*** can be reduced to a second order differential equation of the form:

$$\frac{d^2 A(t)}{dt^2} - cA(t) = 0 \tag{5}$$

the homogenous solution of which has a positive discriminant and real roots, contrary to the claim made in [4]. However, the conclusion reached in [4], that the solution to Lanchester's equation is unstable, still applies although the correct explanation of that instability lies in the discrete nature of the attrition process and the limitations in describing that process with continuous variables [4]. As a result of this instability, real combat outcomes can and should deviate considerably from that predicted by simple application of Lanchester's equations.

Equally, the assumption of "collective" combat is unlikely to apply throughout the entire battle and hence actual attrition results from a combination of collective and individual combats. This possibility has long been recognised and produced many attempts to generalise Lanchester's system of equations to better represent actual combat results. This will also be covered in greater detail below, as will discussion of the assumptions that underlay Lanchester's equations.

A number of these issues were recognised by Lanchester himself, and covered in his writings, through his attempt to validate his system of equations by analysis of the battle of Trafalgar. That analysis has recently been revisited [4] and is worth reviewing here as it illustrates a number of problems encountered in appropriate use of Lanchester's equations.

2.1 Validation

While *real* battles are non-uniform in space and time, neither parameter appears in Lanchester's equations. This results from an implicit assumption that all the forces of each side are able to engage the other (or in other words, the ability to engage is not target limited) in the derivation of the square law.

However, from a study of the evolution of the Battle of Trafalgar, Lanchester realised that a battle is more properly viewed as a series of concurrent and consecutive sub-battles separated by space and time. Lanchester proposed that each of these sub-battles be described using his square law and that the overall casualties of the battle are obtained by summation of those losses. A recent review [4] of this work has shown that each of those sub-battles can also be deconstructed further into a set of smaller sub-battles, and the outcome of the battle can be shown to be strongly dependent on the ability of commanders to fragment the battle to their advantage.

In summary, the successful application of Lanchester's equations to describe actual battle results first requires an understanding of the structure of that battle to enable a hierarchical decomposition of the battle into smaller sub-battles until the assumptions underpinning Lanchester's equations are met by those sub-battles. It could be said that this decomposition results from the application of Command and Control on the battle's evolution. At any rate, the outcomes of those sub-battles can then be determined and the results propagated into the starting conditions for the following group of sub-battles. This process is repeated until the end of the battle. This is the process that Lanchester envisaged, but is not how most practitioners use his theory. While this approach may do better at reproducing the results of known battles, the appropriate deconstruction of a battle into its sub-battles for battles yet to be fought is clearly open to interpretation.

2.2 Interpretation

In the derivation of the square and linear laws, Lanchester interpreted their functional difference as resulting from changes in technology arising from the industrial revolution. His square law was considered to describe "modern" combat while the linear described "ancient" combat.

This is not the interpretation taken by many subsequent practitioners who subscribe to the view that the linear law describes attrition resulting from *indirect* or *area* fire, while the square law describes attrition resulting from *direct* or *point* fire.

However, the work of Karr [5] shows that both the linear and square laws are consistent with both the point and area fire interpretations, which are thus seen to represent *sufficient* but not *necessary* conditions for the derivation of the loss rate terms in equations (1) and (3). This point is so frequently misunderstood by current practitioners that it is worth repeating the description here in full [5].

1. Square Law

- *Point Fire Interpretation* Targets are sufficiently numerous or the ability to locate them is sufficiently good that each attacker locates targets at a constant rate.
- *Area Fire interpretation* Each attacker engages all targets in a certain area per unit time while, targets are dispersed over a region maintaining a constant density so that a reduction in the number of targets reduces the area occupied.

2. Linear Law

- *Point Fire Interpretation* Targets are sufficiently few or sufficiently difficult to locate and attack so that each attacker locates targets at a rate proportional to the number of targets present.
- *Area Fire interpretation* Each attacker engages all targets in a certain area per unit time, while targets are dispersed over a region maintaining a constant area occupied so that a reduction in the number of targets reduces the target density.

Karr [5] also shows that the appropriate assumptions that underlay the derivation of Lanchester's equations are most easily understood in terms of a stochastic Markov Process and will be considered as part of that discussion.

2.3 Heterogeneous Combat Models

The Lanchester equations above use a single force parameter to describe each side's strength. This was a reasonable description of the forces of Lanchester's day. It no longer holds true today, in which each side contains a number of different and effective force elements. A number of approaches to deal with this increased battlefield complexity have been developed.

Heterogeneous Lanchester equations extend equations (1) and (3) to systems of equations where each side's single strength is replaced by a series of strengths describing the numbers of each type of combatant. In this case, the heterogeneous square law equations can be written as:

$$\begin{aligned} \frac{dR_i}{dt} &= -\sum_{j=1}^{N} k_B(i, j) B_j(t) \\ \frac{dB_j}{dt} &= -\sum_{i=1}^{M} k_R(j, i) R_i(t) \end{aligned} \qquad (6)$$

In general, a state equation no longer exists. Similar equations can be developed for the Linear Law. This represents a considerable increase in complexity of the description of the attrition process and a consequent reduction in the model's utility. As such we will not consider the heterogeneous systems of equations further.

Most modern approaches replace this heterogeneous system of equations with an homogenous system of equations in which overall force strength is measured using some form of *force scoring*. Force scoring is a means of aggregating a side's combat strength including both the number of weapons of a given type and an assessment of the relative contribution that each weapon type makes to that side's combat power. Such approaches have been used with varying degrees of success for many years and a number of different force scoring methods have been developed [6] [7].

It is interesting to note that one of these simple deterministic analyses [8] [9] has proven qualitatively better than many of the more sophisticated analyses at predicting the outcomes of recent conventional armed conflicts. This success arises from inclusion of the effect of many factors not considered in the course of development of the Lanchester Equations. Those effects were determined from extensive analysis of historical combat outcomes and are frequently derided as mere curve fitting. Yet the model is simple to use and at least as effective as any other force level combat model.

3. Attrition Metamodels

At its simplest, a metamodel is a model that describes a model. This definition is not as self-referential as it might first appear.

A model is a representation of an actual situation that may be used to better understand that situation. Complex phenomena often require complex models if the models behaviour is to reproduce that of the real world. The modelling of combat attrition is one such situation, as described above [6] [8]. However, while such models produce reasonable agreement with real world results, they are less useful in understanding the functional dependence of the modelled quantity on the input parameters. In such cases it is useful to develop a (simpler) model of that model which, although providing lower fidelity results, is better at explaining the causes of those results. Such models are termed *metamodels*.

Metamodels are often developed using an *ad-hoc* unstructured approach such as dimensional analysis. In recent years a systematic process for the development of such metamodels has emerged that provides a degree of rigour to the undertaking [10].

While the interested reader is referred to the work by Barenblatt [10] for a comprehensive treatment of similarity laws, a brief summary is given below. It must be noted that the use of dimensional analysis in the study of similarity laws is only strictly correct for self-similar problems. However, the distribution of casualties from historical battles have been shown to exhibit self-similar behaviour [11], and hence provides some justification in the application of this approach in combat attrition modelling.

The relationship between an outcome of the model, and a set of input variables can be written as:

$$a = f\left(a_1,...a_k,b_1,...b_m\right) \tag{7}$$

where the variables $a_1,...a_k$ have independent dimensions (the dimension of any of the a's cannot be expressed as a combination of the dimensions of the other a's), and the dimensions of the $b_1,...b_m$ can be expressed in terms of products of powers of the dimensions of $a_1,...a_k$. In general k > 0 and m > 0.

This allows the values of the $a_1,...a_k$ to be independently varied, so that

$$a'_i = A_i a_i \tag{8}$$

The dimensions of a and $b_1,...b_m$ may then be represented as power monomials in the dimensions of $a_1,...a_k$, for example:

$$\begin{aligned}\left[b_j\right] &= \left[a_1\right]^{p_j}...\left[a_k\right]^{r_j}\\ \left[a\right] &= \left[a_1\right]^{p}...\left[a_k\right]^{r}\end{aligned} \tag{9}$$

With the corresponding transformation of values:

$$\begin{aligned}b'_j &= A_1^{p_j}...A_k^{r_j}b_j\\ a' &= A_1^{p}...A_k^{r}a\end{aligned} \tag{10}$$

Introducing the similarity parameters:

$$\Pi_j = \frac{b_j}{a_1^{p_j} \ldots\ldots a_k^{r_j}}$$
$$\Pi = \frac{a}{a_1^{p} \ldots\ldots a_k^{r}} \qquad (11)$$

where the exponents of the variables are chosen so that the parameters Π and $\Pi_1,\ldots\Pi_m$ are dimensionless, enables equation (7) to be re-written as:

$$\Pi = \frac{1}{a_1^p \ldots a_k^r} f\left(a_1,\ldots a_k, \Pi_1 a_1^{p_1} \ldots a_k^{r_1}, \ldots \Pi_m a_1^{p_m} \ldots a_k^{r_m}\right) \qquad (12)$$

Barenblatt [10] shows that this in turn can be re-written in terms of a function of a smaller number of dimensionless variables, leading to the relationship:

$$a = a_1^p \ldots a_k^r \Phi\left(\frac{b_1}{a_1^{p_1} \ldots a_k^{r_1}}, \ldots \frac{b_m}{a_1^{p_m} \ldots a_k^{r_m}}\right) \qquad (13)$$

Furthermore, self-similar solutions correspond to cases where the values of the variables $b_1,\ldots b_m$ tend to zero or infinity. Barenblatt [10] considers three cases:

1. Type 1 metamodel.

The function Φ tends to a non-zero finite limit as Π_j tends to zero or infinity. In practice this means Φ can be replaced by its limiting expression, and hence *f* will be a product of power monomials whose values can be determined by dimensional analysis.

2. Type 2 metamodel.

The function Φ tends to the power law asymptotic expression:

$$\Phi = \Pi_j^{\alpha} \Phi\left(\frac{\Pi_1}{\Pi_j^{\alpha_1}}, \ldots \frac{\Pi_m}{\Pi_j^{\alpha_m}}\right) \qquad (14)$$

as Π_j tends to zero or infinity. The power law form of the limiting expression still leads to complete separation of variables, but with characteristic exponents which, in contrast to the type 1 metamodels, cannot all be determined by dimensional analysis.

3. Type 3 metamodel.

Power type asymptotic behaviour is not observed, and the function Φ has no finite limit different from zero.

4. Generalised Lanchester Equations

As observed above, Lanchester's equations have generally failed to correspond with available historical data [3]. More limited examination of the historical data has found related sub-sets of that data (such as the Ardennes Campaign and the Iwo Jima Campaign) that are consistent with the different forms of Lanchester's laws [3]. This has lead many researchers to view Lanchester's original equations as special cases of a more general attrition equation, or as attrition metamodels as discussed in the previous section. Several different means for obtaining that more general attrition equation, or Generalised Lanchester equation, have been proposed. They fall into two basic categories, depending on their fundamental paradigm.

4.1 Deterministic Approaches

There are two distinct approaches for deterministic generalised Lanchester equations: arbitrary power law formulations and non-monotonic formulations.

Non-monotonic formulations replace the Lanchester equations of equation (1) with equations of the form:

$$\begin{aligned} \frac{dR}{dt} &= A - k_B B(t) - q_R R(t), \quad R(0) = R_0 \\ \frac{dB}{dt} &= B - k_R R(t) - q_B B(t), \quad B(0) = B_0 \end{aligned} \qquad (15)$$

where the additional terms can represent reinforcements and decision thresholds, such as in the Dewar model [12], or production as in the strategic conflict models described by Kimball [13]. These models can result in highly non-monotonic, even chaotic, behaviour [8] that does describe some features of combat attrition not seen in Lanchester's laws. However, they are in general no more successful in replicating the broad range of available historical data than the original Lanchester's equations.

The more common approach is to regard Lanchester's equations as metamodels resulting from the application of equation (13) above to a generalised Lanchester equation, where the assumptions and constraints applied to the generalised equation have specified the values of the monomial exponents. The general expressions then have the form:

$$\begin{aligned} \frac{dR}{dt} &= -k_B^C B(t)^D R(t)^E, \quad R(0) = R_0 \\ \frac{dB}{dt} &= -k_R^F R(t)^G B(t)^H, \quad B(0) = B_0 \end{aligned} \qquad (16)$$

where the exponents (*C..H*) are not constrained to integer values, but usually $D\text{-}G = H\text{-}E$ and $G+H = D+E = 1$. This general expression does not have to be symmetric in *B(t)* and

R(t). It is also interesting to note that the same equation of state results from different values of the exponents if these two constraints are met.

Substantial work has been done using this form of the generalised Lanchester equations by a number of researchers including Hartley [3] and Helmbold [14], to which the interested reader is referred. If *D-G* = *H-E* = α-1 and *C-F* = β, an equation of state can be found for the system of equation (16) such that [14]:

$$\ln\left(\frac{R_0^2 - R(t)^2}{B_0^2 - B(t)^2}\right) = \alpha \ln\left(\frac{R_0}{B_0}\right) + \beta$$
$$\ln(HelmboldRatio) = \alpha \ln(ForceRatio) + \beta \qquad (17)$$

Hartley [3] has analysed the results of over 600 battles and graphed the parameters in the equation of state, as shown in Figure 1.

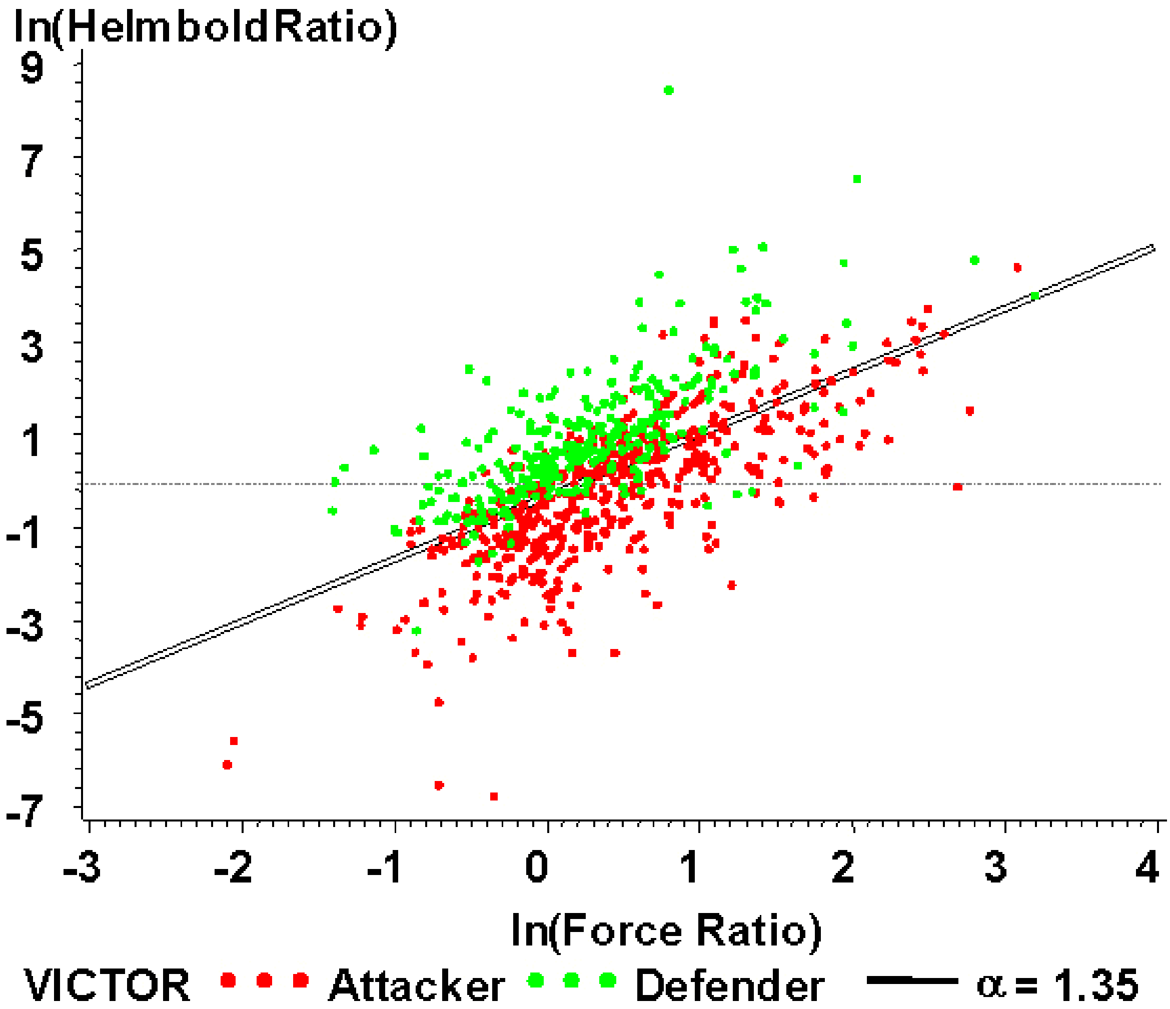

Figure 1: Helmbold's Relationship, from Hartley [3].

The line of best fit indicates the relationship between initial and final strengths that would apply if Lanchester like attrition was the sole process at work. The variation of outcomes with respect to that relationship is due to variation in the Lanchester attrition coefficients with factors such as cohesion, leadership, terrain and weather that are not explicitly dealt with in his equations. There have been a number of attempts to determine the coefficient's dependence on those properties such as the Quantified Judgement Model (QJM) [6] and the Oak Ridge Spreadsheet Model (ORBSM) [3]. Most attempts, however, have focussed on *a-priori* calculation of the coefficient from first principles using the flawed interpretations of the form of Lanchester's laws in representing point and area fire already referred to.

4.2 Stochastic Approaches

A continuous time, discrete space, Markov process can lead to stochastic attrition models analogous to the deterministic Lanchester equations in a rigorous manner. This has been comprehensively explored by a number of workers and is readily accessible [13] [15]. The treatment summarised here follows that of Karr [5].

If X is a regular continuous time Markov process with countable space E = N^2 the set of integer pairs (i,j) with both (i,j) ≥ 0, then:

$$\begin{aligned} X &= (X_t)_{t\geq 0} \\ P_t(x,y) &= P_x\{X_t = y\} \\ P_{t+s}(x,y) &= \sum_{z\varepsilon E} P_t(x,z)P_s(z,y) \end{aligned} \tag{18}$$

where P_x is the probability that the initial state (t = 0) is x. The infinitesimal generator matrix A is then defined by:

$$\begin{aligned} A(x,y) &= \lim_{h\to 0}\frac{P_h(x,y)-I(x,y)}{h} \\ P\{X_{t+h} = y | X_t = x\} &= A(x,y)h + o(h) \\ P_t' &= P_t A \end{aligned} \tag{19}$$

A Markov process (B_t, R_t) with state space E can be regarded as a combat attrition model between a homogenous R side and a homogenous B side provided that the paths t → (B_t, R_t) are non-increasing. Here both B_t and R_t are random variables for each t. Equation (19) is then analogous to the Lanchester attrition equations depending on the derivation of an appropriate generator matrix A.

Karr [5] describes a Markov process with the following assumptions:

1. All combatants on each side are identical (homogenous).

2. Times between detections by a surviving red combatant are independent and identically exponentially distributed with mean r_1^{-1}, regardless of the (non-zero) number of surviving blue combatants.
3. When a red combatant detects a blue combatant an instantaneous attack occurs, in which the blue combatant is destroyed with probability p_1 and survives with probability $1 - p_1$. Total loss of contact immediately takes place.
4. Blue combatants satisfy assumptions 2 and 3 with parameters r_2 and p_2 respectively.
5. Conditioned on survival, detection and attack processes of all combatants are mutually independent (in the probabilistic sense).

This can be described by the following generator matrix:

$$\begin{aligned} A((i,j),(i-1,j)) &= k_1 j \\ A((i,j),(i,j-1)) &= k_2 i \\ A((i,j),(l,m)) &= 0 \ldots \forall \text{ other pairs of } (i,j),(l,m) \\ A((i,j),(i,j)) &= -(k_1 j + k_2 i) \end{aligned} \qquad (20)$$

Where the assumptions above define the loss rate:

$$k_i = r_i p_i \ldots i = 1,2 \qquad (21)$$

This is recognisable as analogous to Lanchester's Square Law. A similar treatment, albeit with slightly different assumptions, results in Lanchester's Linear Law [5].

Karr goes on to show that the expectation values of the forces' strengths do not in general satisfy the deterministic Lanchester Equations in the square law case. However, given the additional assumption that the loss rates are small, an equation of state for the continuous time Markov process can eventually be found from:

$$\frac{d}{dt} E\left[k_2\left(B_t^2 + B_t\right) - k_1\left(R_t^2 + R_t\right)\right] = 0 \qquad (22)$$

5. Fractal Attrition Equations

Lauren's combat attrition metamodel [1] was developed from a description of the attrition process in the MANA cellular automata combat model. The key process underlying the model is the means by which agents from either side discover opposition agents. This is a probabilistic process and hence results in a discrete time stochastic model.

During a time step of duration Δt, an agent travelling at a velocity v will travel a distance $d = v\Delta t$. The agent is then able to search an area, for simplicity assumed to be a square box with dimension d, for opposition agents during that time interval. If the opposition force is treated as consisting of clusters of agents, and the clusters have a fractal spatial distribution described by a fractal dimension D, then the probability that at least one opposition agent will be encountered in that box is proportional to d^{-D}. If the average size of a cluster is n_d, then the average number of opposition agents encountered is proportional to $d^{-D}n_d$. If $R(t)$ is the number of opposition agents and $\varphi(x,t)$ is the number of clusters, then the expectation value for the number of agents killed by opposition agents during that time step is:

$$E\left(\frac{\Delta B}{\Delta t}\right) \propto k(\Delta t)^{2-D} R(t)\varphi(x,t) \tag{23}$$

Invoking Barenblatt's metamodelling approach [10] above, this can be re-written as:

$$E\left(\frac{\Delta B}{\Delta t}\right) = f\left(k, (\Delta t)^{2-D}, R(t)\varphi(x,t)\right) \tag{24}$$

Relaxing the assumptions that were used to obtain this general attrition equation, applying Barenblatt's asymptotic expansion process and constraining the values of the exponents with dimensional analysis, leads to the result:

$$E_{0<t<T}\left(\frac{\Delta B}{\Delta t}\right) \propto k^{\alpha} \Delta t^{\alpha-1} R(t) \tag{25}$$

where $\alpha = D/2$. The case $\alpha = 1$ can be shown to correspond with the assumptions of Lanchester's Square Law. Equation (25) then reduces to the expected Lanchester result.

This model is usually specified in terms of the Fractal Attrition Equation (FAE):

$$E_{0<t<T}\left(\frac{\Delta B}{\Delta t}\right) = E_{0<t<T}\left(-ck^{\frac{D}{2}} \Delta t^{-(1-\frac{D}{2})} R(t)\right) \tag{26}$$

5.1 Interpretation

If the combat system is assumed to be self-similar, then *multifractal statistics* apply. This means that the form of expression above applies in an analogous way to all of the moments of the random variable. Thus we have:

$$E_{0<t<T}\left(\left|\frac{B(t+\Delta t) - B(t)}{\Delta t}\right|^{p}\right) \propto \Delta t^{g(D,p)} \tag{27}$$

In particular, consideration of the Fourier transform of the second moment of the random variable yields a power spectrum for the frequency distribution of a side's casualties:

$$\left|\Im(B)(f)^2\right| \propto |f|^{-(D+1)} \tag{28}$$

In general, the precise form of the power law exponent has to be determined experimentally. Examination of available historical data is consistent with this relationship [16].

5.2 Fractal Dimension

The introduction of a fractal dimension to describe the spatial distribution of a side's units in equation (23) was empirical. That is, there was no *a-priori* quantification of how that distribution influenced the attrition process, just the qualitative observation that the available historical data was consistent with that assumption. It remains, therefore, to describe how a fractal dimension can be specified that describes the differences between different spatial distributions of units in combat situations.

A number of different techniques for the measurement of a fractal dimension have been developed [17]. It is by no means clear which is the more appropriate or accurate. However, given the close relationship between the FAE metamodel and the MANA cellular automata simulation, a box counting technique that is appropriate to MANA was chosen [1]:

$$D = \lim_{d \to 0} \frac{\ln N}{\ln(1/d)} \tag{29}$$

where N is the number of boxes necessary to cover all the automata.

5.3 Validation

MANA [11] is different from traditional cellular automata models in that it includes limited global interactions between automata as well as local interactions. The automata's behaviours are governed by sets of parameters that determine whether they move towards friendly or enemy units, towards an objective and their interactions with other automata. Additional parameters allow event driven conditional modifiers to these behaviours, such as requiring certain conditions to be met before an automaton can move towards an objective or foe. A final set of parameters describes the basic capabilities of the automaton, such as weapons range, detection range and movement rate.

Global interactions are facilitated by providing each side a map on which the locations of enemy automata can be stored. The automata react to both units that are detected directly and those they are aware of on the map. The map thus simulates a rudimentary communications network.

MANA automata use a penalty function to rank possible moves based on their rules set. If several moves have a similarly low penalty value, one of them is chosen at random. A parameter sets the size of the margin for discriminating between similarly valued moves.

Interactions between automata allow complex combat behaviours to be simulated, including:

1. The automata of each side are attracted towards a goal behind its opponent.
2. Initially, neither side is in contact with the other.
3. No automaton may advance without the presence of at least some friendly automatons within detection range.
4. Once enemy automata are detected, a minimum number of friendly automata is required within detection range to continue advancing.
5. An automaton that is unable to advance will retreat.
6. Once no enemy are within detection range, retreating automata may regroup and advance again.

Hence MANA is a discrete time stepped stochastic simulation that allows concentration and reorganisation of units in both space and time. Attrition results from consecutive stochastic detection and engagement processes. The continuous reorganisation of the forces dispositions has similarities with withdrawal and reinforcement of combat forces. The processes that underlie this reorganisation result from a combination of local knowledge and Command and Control (C2) effects from use of the map mechanism. The relationship between the MANA simulation's design and the assumptions made in developing the FAE should be immediately apparent.

Comparing the results of MANA simulations of a number of scenarios with the implications of the FAE metamodel has been undertaken [1]. Comparing a metamodel with a simulation is straightforward as, in principle, all the parameters in the simulation can be systematically varied (such as k). This enables the full parameter space to be explored for consistency with the metamodel. These comparisons have suggested that, as a general rule, Lanchester's Square Law was more accurate than the FAE metamodel when the automata maintained a single clump-like formation, or when there was a disparity in k between the two sides. In all other cases the FAE metamodel was more accurate.

Such comparisons with historical data are much more difficult. For example, k is then an artefact of the experimental data chosen and cannot be independently varied.

Beginning with the FAE, the loss exchange ratio (LER) for both sides can be obtained:

$$\frac{\langle B(t+\Delta t)-B(t)\rangle}{\langle R(t+\Delta t)-R(t)\rangle}=\frac{(k_R\Delta t)^{D_R/2}R(t)}{(k_B\Delta t)^{D_B/2}B(t)} \tag{30}$$

Given the assumption that the entities have a fractal distribution with cell size d being chosen so that each cell contains at most one entity, then:

$$R\propto\frac{1}{d^D}\Rightarrow D=c-\log_d R \tag{31}$$

is in general true and can thus be evaluated at time t = 0. Further, if the constants relating to the red and blue forces are similar, the equation of state:

$$\frac{|\Delta B|}{|\Delta R|} \propto \left(\frac{B_0}{R_0} \right)^Q \tag{32}$$

can be obtained and used in comparison with the available historical data. The exponent Q is as defined in [1].

6. The Fractal Attrition Equation; a Better Representation?

The FAE has been derived using the assumptions and constraints that describe the operation of the MANA simulation. It has been shown to produce results that are consistent with those of the MANA simulation [1]. To all effective purposes, the FAE has been verified as providing a suitable metamodel for the MANA simulation.

The assumption of a fractal spatial distribution for each side's forces was shown to result in a fractal temporal distribution for casualties, in keeping with much of the available historical data. Furthermore, Lanchester's Square Law is shown to be the limiting case for the FAE on application of Lanchester's additional assumptions.

Reference [1] suggests that the FAE metamodel could prove a more accurate representation of combat than the original Lanchester equations, because it includes factors that are relevant to detection/manoeuvre as well as to firepower. This issue is explored further in this section.

6.1 Dependent vs. Independent Dimensions

Barenblatt's metamodelling technique emphasises the difference between dependent and independent variables. In this context, an independent variable is a variable whose dimension cannot be expressed as a combination of the dimensions of the other independent variables. The asymptotic limits of independent and dependent variables are treated quite differently, as seen in equation (13).

The FAE model is a two step process involving detection followed immediately by combat. The detection process is stochastic and involves the distance $\boldsymbol{d = v\Delta t}$. However, in the derivation this is replaced immediately with the time interval parameter, resulting in the generalised attrition equation (24). This is where the difficulty occurs. Equation (24) has two parameters ($\boldsymbol{k, \Delta t}$) that have time as their dimension. Only one of them should be treated as a power law monomial in the asymptotic approximation, in accordance with Barenblatt's approach [10] and equation (13). Equation (24) was derived after this relationship between space and time ($\boldsymbol{d = v\Delta t}$) was defined. What should have occurred was that this constraint be applied after the asymptotic approximation was made. Hence, equation (23) should be replaced with:

$$E\left(\frac{\Delta B}{\Delta t}\right) = f\left(k, d^{-D}, R(t)\varphi(x,t)\right) \qquad (33)$$

which has suitable independent variables. Application of the intermediate asymptotic technique yields in place of equation (25):

$$E_{0<t<T}\left(\frac{\Delta B}{\Delta t}\right) \propto k^{\alpha} d^{\beta} R(t) \qquad (34)$$

The relationship between space and time intervals can now be applied, with the velocity term being absorbed into the proportionality constant and dimensional analysis used to constrain the values of α and β. This results in the FAE of equation (26) as before. Hence this minor inconsistency in derivation does not affect the FAE model's results.

6.2 To Be or not To Be Lanchester's Equations

Reference [1] argues that the FAE is sufficiently different to Lanchester's Square Law, on a number of grounds, that it warrants consideration as a replacement for Lanchester's Equation. Among these differences, the FAE incorporates a measurement of the effect of the spatial distribution of forces and the consequent Loss Exchange Ratio (LER) exhibits different behaviours in time. However, reference [1] also shows that Lanchester's Square Law is the limiting case for the FAE, given a fractal dimension of 2 corresponding to a uniform force distribution., and suggests that the fractal dimension of the spatial distribution of forces might be a measure of the state of its Command, Control, Communications, Computers, Intelligence, Surveillance and Reconnaissance (C4ISR) effectiveness.

As previously mentioned, many subsequent users of Lanchester's theory have failed to understand how it was intended to be used. In particular, Lanchester realised that a battle is more properly viewed as a series of concurrent and consecutive sub-battles separated by space and time. Lanchester proposed that each of these sub-battles be described using his square law and that the overall casualties of the battle are obtained by summation of those losses. Hence Lanchester intended that the spatial and temporal distribution of forces be taken into account in the application of his theory. He just felt that it was appropriate to do this manually with a discrete number of sub-battles corresponding to identifiable phases in an overall engagement. The success of the FAE is to describe a process, for a wide range of situations, in which this *ensemble averaging* of sub-battles can be analytically incorporated into the attrition equations themselves. This is a significant addition to Lanchester's theory, but does not by itself invalidate that theory. Indeed, it seems just as likely that the FAE may be no more than another of the class of generalised Lanchester Equations corresponding to a different set of assumptions.

The similarities do not end here. In the limit as the time interval tends to zero, the FAE of equation (26) can be written more generally as:

$$\begin{aligned} \frac{dR}{dt} &= -k_B G(t)B(t), \quad R(0) = R_0 \\ \frac{dB}{dt} &= -k_R G(t)R(t), \quad B(0) = B_0 \end{aligned} \qquad (35)$$

Where G(t) describes the time dependence of the attrition rate and can be considered a measure of how the intensity of combat changes over time. The equations have, for simplicity, been assumed symmetric for blue and red forces. In the FAE this behaviour is the reason why the LER can remain a constant, in spite of changing force ratios, and in agreement with the MANA simulation [1].

However, Koopman [18] has shown that the time scale is a rather arbitrary parameter in Lanchester's theory and can be varied in accordance with Barenblatt's gauge transformation [10] technique, without changing the underlaying functional relationships [19]. Therefore, introducing a new time scale τ such that

$$\tau = c\int_0^t G(t)dt \qquad and \qquad \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} \qquad (36)$$

which on substitution into equation (35) yields:

$$\begin{aligned} \frac{dR}{d\tau} &= -\frac{k_B}{c}B(\tau), \quad R(0) = R_0 \\ \frac{dB}{d\tau} &= -\frac{k_R}{c}R(\tau), \quad B(0) = B_0 \end{aligned} \qquad (37)$$

This is of course Lanchester's Square Law relationship. The transformation of equation (36) can be interpreted as a "*linearisation*" of the combat in time. The combat then has a constant intensity when measured against time τ, but varies in intensity against time t. The explicit time independence of the attrition process is one of Lanchester's assumptions, so it is useful to note that the FAE reduce to this form when the time dependence of the combat is incorporated into the definition of the time variable itself instead of being left explicit.

However, it does mean that the FAE are a form of Lanchester's Equations.

6.3 Stochastic Lanchester Equation

The treatment of Lanchester attrition as a continuous time Markov process is described in equations (18) to (21) above. The attrition loss rate is written as a product of two terms; the probability of detection of an enemy and the probability of destruction of a detected enemy. The particular form of the probability for detection depends on the model behind the detection process. Karr [5] illustrates the process for an exponentially distributed probability of detection.

The FAE was developed using a different set of assumptions, in which the battlefield of area A is divided into a number of boxes (with dimension d) that are systematically searched in turn. This model is described by a geometric probability distribution with a constant probability of a friendly unit finding an enemy unit in any given box of r. The resulting mean time between detections follows from the geometric probability distribution as $1/r$.

A Markov attrition process, with a geometric probability distribution describing the detection process, results in the same set of equations (18) to (21) except that the attrition parameters (equation (21)) are obtained using the geometric probability distribution values. It just remains to determine the probability for finding an enemy unit in any given box in this model.

If there are N boxes and the probability of occupation for a box is p, then for a geometric probability distribution, the mean number of occupied boxes is Np. If the force has a fractal spatial distribution described by a fractal dimension D, then the number of occupied boxes is given by d^{-D}. This results in the probability that an enemy unit is detected in any given box of:

$$p = \frac{1}{Nd^{D}} = \frac{d^{2-D}}{A} \tag{38}$$

which, on remembering that this model assumes $d = v\Delta t$, and substituting into equation (21) yields an expression consistent with equation (23), the FAE precursor expression. Hence the FAE can be derived as a stochastic Lanchester equation, given the assumptions that the detection process has a geometric probability distribution and a fractal spatial distribution. The FAE, equation (25), might just as well be called the Fractal Lanchester Equation.

6.4 Fractal Equation of State

It has been demonstrated above, that for each of the attrition expressions corresponding to the different forms of Lanchester's Square Law examined above, there is an analogous equation of state.

Table 1: Model Correspondences

Model	Attrition Equation Reference	Equation of State Reference
Lanchester Square Law	(1)	(2)
Generalised Lanchester	(16)	(17)
Stochastic Lanchester	(18)	(22)
FAE	(26)	(32)

It should also be noted that when the assumptions behind the alternate forms of Lanchester's Square Law were relaxed to agree with those of Lanchester's Square Law, the limiting form of the corresponding attrition expression also agreed with Lanchester's Square Law.

This agreement with Lanchester's Square Law as the limiting case should also occur for the corresponding equations of state, after all the attrition expressions from which they are derived agree in this limit. Agreement was observed for the Generalised Lanchester model and the Stochastic Lanchester model, once the additional assumption of large forces that was made in its derivation [5] is understood.

This agreement in the limit is not observed for the original FAE Equation of State of reference [1], equation (32). The Lanchester Square Law Equations of State are expressions that involve the square of the forces' strengths, hence the name. However, the FAE Equation of State, resulting from its Square Law attrition expression, is linear in the forces' strengths.

Examining the derivation of equation (32) in [1], it was seen that the spatial resolution of the battlefield $\boldsymbol{d}$ was chosen as if it were independent of the time interval $\boldsymbol{\Delta t}$, in the derivation of an approximation expression for the fractal dimension $\boldsymbol{D}$. This is in contrast to the derivation of the attrition equation (26) where the relationship between the time interval and spatial resolution was fundamental to the form of the FAE. The consequence of this inconsistency was that the time interval was varied independently of the spatial resolution, and hence was used to relate the initial and final strength values in the equation of state resulting in its linear form.

The resolution of this inconsistency comes from recognition that specifying the spatial resolution, and hence the fractal dimension, also determine the time interval. If we take d to be the length of the cell of the smallest area which at most contains a single Red entity, and if we assume that d and k are about the same for the Red and Blue forces, substituting equation 31 into the right hand side of equation (30) produces:

$$\frac{R(t)}{B(t)}(k\,\Delta\,t)^{\log_d(R_0/B_0)/2} \tag{39}$$

which can be rearranged to give:

$$\frac{R(t)}{B(t)}\left(\frac{B_0}{R_0}\right)^Q \tag{40}$$

where Q has the form:

$$Q = 0.5\log_d(k\,\Delta t) \tag{41}$$

Replacing the right hand side of equation (30) with equation (40), and cross-multiplying yields:

$$B(t)\Delta B = \left(\frac{B_0}{R_0}\right)^Q R(t)\Delta R \qquad (42)$$

Dividing the interval between the beginning and ending of the engagement into a series of time steps allows equation (40) to be applied to each of these time steps. A connection between the initial and final force strengths can then be obtained by summing all those expressions:

$$\sum_{\text{int}\, ervals} B\Delta B = \left(\frac{B_0}{R_0}\right)^Q \sum_{\text{int}\, ervals} R\Delta R \qquad (43)$$

In keeping with the development of the FAE from a metamodel for MANA, the force strengths represent the number of live entities. When attrition occurs, that number is reduced. This suggests that a stochastic model is more appropriate. If we now apply this additional assumption, that attrition is modelled as a Markov attrition process of the type examined by Karr [5], the probability of a change in the state (force strength) can be obtained from equation (19). The form of the infinitesimal generator matrix only allows transitions to states whose strengths differ from the initial state by 1. The assumption of a small Δt can be interpreted as equivalent to the probability of more than one transition in any given time interval being small, and which in the first instance can be neglected. Remembering the connection between the time interval and spatial resolution, the time interval Δt can be chosen small enough such that its corresponding spatial resolution also satisfies the requirements of equation (31). Then in any interval in the sum terms of equation 43 in which there is no state transition, either ΔB or ΔR is 0 and equation (43) reduces to:

$$\sum_{B=B_0}^{B(t)} B = \left(\frac{B_0}{R_0}\right)^Q \sum_{R=R_0}^{R(t)} R \qquad (44)$$

Evaluating the sums, results in:

$$\ln\left(\frac{\left(B^2(t) + B(t)\right) - (B_0^2 + B_0)}{\left(R^2(t) + R(t)\right) - (R_0^2 + R_0)}\right) = Q\ln\left(\frac{B_0}{R_0}\right) \qquad (45)$$

This has obvious similarities to the equations of state from both the Stochastic and Generalised Lanchester models, and has Lanchester's Square law as the limiting case as required. It is also in agreement with the comprehensive historical data of Figure 1. Furthermore, by relaxing the assumption that the value of k is about the same for the Red and Blue forces, and setting $k_R = rk_B$ in equation (30), the equation of state can be re-evaluated as:

$$\ln\left(\frac{\left(B^2(t)+B(t)\right)-(B_0^2+B_0)}{\left(R^2(t)+R(t)\right)-(R_0^2+R_0)}\right)=Q\ln\left(\frac{B_0}{R_0}\right)+0.5D_R\ln r \qquad (46)$$

The similarity between this equation and the relationship developed by Helmbold (equation (17) above) to describe his analysis of real-world air and land combat results, and used by Hartley in his work on combat modelling (in Figure 1), is obvious. In the cases considered by Helmbold and Hartley, the forces strengths are sufficiently large such that the difference between B^2 and $B(B+1)$ would not be observed.

This last expression is proposed as the correct equation of state for the FAE model, as a replacement for equation (32). This was accepted at the TTCP JSA TP3 workshop held at Fishermans Bend in November 2005.

7. Conclusions

The FAE has been derived using the assumptions and constraints that describe the operation of the MANA simulation [1]. Furthermore, it has been shown to produce results that are consistent with those of the MANA simulation. For all effective purposes, the FAE has been verified as providing a suitable metamodel for the MANA simulation.

The assumption of a fractal spatial distribution for each side's forces was shown to result in a fractal temporal distribution for casualties, in keeping with much of the available historical data. Furthermore, Lanchester's Square Law was shown to be the limiting case for the FAE on application of Lanchester's additional assumptions.

The FAE uses a novel approach to incorporating the spatial and temporal variations in combat intensity into a single *force-on-force* equation. However, this does not represent a radical departure from traditional practice. Lanchester himself proposed a paradigm that involved manually dividing the battlefield into sub-battles and aggregating their results. Indeed, it is easy to show by defining a new time variable that the FAE reduces to Lanchester's form.

Furthermore, the FAE can be derived as a stochastic Lanchester equation, given the assumptions that the detection process has a geometric probability distribution and a fractal spatial distribution. The FAE might just as well be called the Fractal Lanchester Equation.

Lastly, it is shown that the original FAE equation of state was derived using mutually exclusive constraints on the relationship between the spatial and temporal resolution applied to the battlefield. A new equation of state, using a consistent interpretation of that relationship, is derived as a replacement for the proposed FAE equation of state. This equation is not linear in the forces' casualty ratio, as is the original FAE result. It describes the ratio of the difference in the square of the forces' initial and final strengths. This new relationship is consistent with both the deterministic and stochastic forms of Lanchester's

Square Law. It is also consistent with the historical data analysis due to Helmbold and presented in Figure 1 above.

As a direct result of the present work, the July 2005 draft of the TTCP report [1] has been revised to include the FAE Equation of State derived here.

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19. ABSTRACT

Under the auspices of The Technical Cooperation Program (TTCP) Joint Systems & Analysis (JSA) Technical Panel 3 (Joint Concepts and Analysis) research program, researchers in New Zealand and the United Kingdom have been collaborating on the development of a new combat attrition equation. This report examines this new approach, its relationships to existing systems of Lanchester equations and its prospects for further development.

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